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Invariant bilinear forms on a vertex algebra

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Abstract

In this paper we construct a linear space that parameterizes all invariant bilinear forms on a given vertex algebra with values in a arbitrary vector space. Also we prove that every invariant bilinear form on a vertex algebra is symmetric. This is a generalization of the result of Li (J. Pure Appl. Algebra 96(3) (1994) 279), who proved this for the case when the vertex algebra is non-negatively graded and has finite dimensional homogeneous components.

As an application, we introduce a notion of a radical of a vertex algebra. We prove that a radical-free vertex algebra A is non-negatively graded, and its component A_0 of degree 0 is a commutative associative algebra, so that all structural maps and operations on A are A_0-linear. We also show that in this case A is simple if and only if A_0 is a field.

MSC: 17B69

0. Introduction

Invariant bilinear forms on vertex algebras have been around for quite some time now. They were mentioned by Borcherds in [1] and were used in many early works on vertex algebras, especially in relation with the vertex algebras associated with lattices [2,6,10]. The first systematic study of invariant forms on vertex algebras is due to Frenkel et al. [5]. This theory was developed further by Li [12].

However, these authors imposed certain assumptions on their vertex algebras which are too restrictive for the applications we have in mind. Specifically, this paper is motivated by the study of vertex algebras of OZ type generated by their Griess subalgebras [8,17]. We show that all the results of Li [12] hold in the greatest possible generality,

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basically, as long as the definitions make sense. We construct a linear space that parameterizes all bilinear forms on a given vertex algebra with values in a arbitrary linear space, and also prove that every invariant bilinear form on a vertex algebra is symmetric. Our methods, however, are very different from the methods used in [5] and [12].

As an application, we introduce a very useful notion of radical of a vertex algebra. It is equal to the radical of a certain canonical invariant bilinear form. We prove that a radical-free vertex algebra A is non-negatively graded, and its component A_0 of degree 0 is a commutative associative algebra, so that all structural maps and operations on A are A_0-linear. We also show that in this case A is simple if and only if A_0 is a field.

The paper is organized as follows. In Section 1 we give definitions of some standard notions of the theory of vertex algebras, including modules and sl_2-structure. Then in Section 2 we construct the universal enveloping algebra $U(A)$ of a vertex algebra A and prove that it has a canonical anti-involution. Our enveloping algebra $U(A)$ is a subalgebra of the universal enveloping algebra constructed in [7], this allows us to deal with a more general type of modules. The existence of anti-involution on $U(A)$ can be derived from the results of [5].

In Section 3 we prove the main result of this paper: we describe all invariant bilinear forms on a vertex algebra A. In Section 4 we introduce a notion of a radical of a vertex algebra and investigate radical free-algebras. Finally, in Section 5 we discuss some examples.

1. Definitions and notations

Here we fix the notations and give some minimal definitions. For more details on vertex algebras the reader can refer to the books [6,10,11]. All spaces and algebras are considered over a field $\Bbbk$ of characteristic 0. We use the following notation for divided powers:

$$x^{(n)} = \begin{cases} (x^n)/n! & \text{if } n \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

1.1. Definition of vertex algebras

Definition 1. A vertex algebra is a linear space A equipped with a family of bilinear products $a \otimes b \mapsto a(n)b$, indexed by integer parameter n, and with an element $\mathbb{1} \in A$, called the unit, satisfying identities (i)–(iv) below. Let $D : A \to A$ be the map defined by $Da = a(-2)\mathbb{1}$. Then

(i) $a(n)b = 0$ for $n \gg 0$,
(ii) $\mathbb{1}(n)a = \delta_{n,-1}a$ and $a(n)\mathbb{1} = D^{(-n-1)}a$,
(iii) $D(a(n)b) = (Da)(n)b + a(n)(Db)$ and $(Da)(n)b = -na(n-1)b$,
(iv) $a(m)(b(n)c) - b(n)(a(m)c) = \sum_{s \geqslant 0} \binom{m}{s} (a(s)b)(m+n-s)c$

for all $a, b, c \in A$ and $m, n \in \mathbb{Z}$.

This is not the only known definition of vertex algebras. Often the axioms are formulated in terms of the *left adjoint action map* (a.k.a. *state-field correspondence*) $Y : A \to A[[z,z^{-1}]]$ defined by $Y(a)(z)=\sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$, where $a(n): A \to A$ is the operator given by $b \mapsto a(n)b$. The most important property of these maps is that they are *local*: for any $a,b\in A$ there is N such that

$$[Y(a)(w), Y(b)(z)](w-z)^N = 0.$$

The minimal $N=N(a,b)$ for which this identity holds is called *the locality index* of a and b. In fact, $N(a,b)=\min\{n\in\mathbb{Z} \mid a(m)b=0\ \forall m \geqslant n\}$. From (iii) we get $Y(Da)(z)=\partial_z Y(a)(z)$, which in terms of coefficients means

$$[D, a(m)] = -ma(m-1). \tag{1}$$

Among other identities that hold in vertex algebras are the quasi-symmetry

$$a(n)b = -\sum_{i\geqslant 0}(-1)^{n+i}D^{(i)}(b(n+i)a) \tag{2}$$

and Borcherds identity

$$\begin{aligned}\sum_{s\geqslant 0}\binom{k}{s}&(a(m+s)b)(n+k-s)c\\ &=\sum_{s\geqslant 0}(-1)^s\binom{m}{s}a(m+k-s)(b(n+s)c)\\ &\quad-\sum_{s\leqslant m}(-1)^s\binom{m}{m-s}b(n+s)(a(m+k-s)c)\end{aligned} \tag{3}$$

for any $k,m,n\in\mathbb{Z}$. For $k=0$, Borcherds identity gives the so-called associativity identity

$$\begin{aligned}(a(m)b)(n)c=&\sum_{s\geqslant 0}(-1)^s\binom{m}{s}a(m-s)(b(n+s)c)\\ &-\sum_{s\leqslant m}(-1)^s\binom{m}{m-s}b(n+s)(a(m-s)c).\end{aligned} \tag{4}$$

For $m\geqslant 0$ this simplifies to

$$(a(m)b)(n)c=\sum_{s=0}^{m}(-1)^s\binom{m}{s}[a(m-s),b(n+s)]c$$

which is just another form of identity (iv) of Definition 1.

A vertex algebra A is called graded (by the integers) if $A=\bigoplus_{d\in\mathbb{Z}}A_d$ is a graded space, $A_i(n)A_j\subseteq A_{i+j-n-1}$ and $\mathbb{1}\in A_0$.

It is often required that a vertex algebra A is graded and A_2 contains a special element ω such that $\omega(0)=D$, $\omega(1)|_{A_d}=d$ and the coefficients $\omega(n)$ generate a representation of the Virasoro Lie algebra:

$$[\omega(m), \omega(n)] = (m-n)\omega(m+n-1) + \delta_{m+n,2}\,\frac{1}{2}\begin{pmatrix} m-1 \\ 3 \end{pmatrix} c$$

for some constant $c \in \Bbbk$ called the *central charge* of A. In this case A is called *conformal vertex algebra* or *vertex operator algebra*, especially when $\dim A_d < \infty$.

1.2. *The action of* sl_2

In order to work with bilinear forms, we need to deal with vertex algebras equipped with certain additional structure. First of all we will assume that our vertex algebra $A = \bigoplus_{d\in\mathbb{Z}} A_d$ is graded. We will also need a locally nilpotent operator $D^* : A \to A$ of degree -1, such that $D^*\mathbb{1} = 0$ and

$$[D^*, a(m)] = (2d - m - 2)a(m+1) + (D^*a)(m) \tag{5}$$

for every $a \in A_d$. Let $\delta : A \to A$ be the grading derivation, defined by $\delta|_{A_d} = d$. It is easy to compute that

$$[D^*, D] = 2\delta, \quad [\delta, D] = D, \quad [\delta, D^*] = -D^* \tag{6}$$

so that D^*, D and δ span a copy of sl_2.

An element $a \in A$ such that $D^*a = 0$ is called *minimal*. It is easy to see that if A is generated by minimal elements, then any operator $D^* : A \to A$ satisfying (5) must be locally nilpotent. For any $a \in A$ define $\operatorname{ord} a = \min\{k \in \mathbb{Z}_+ \mid (D^*)^{k+1}a = 0\}$, so that $\operatorname{ord} a = 0$ for a minimal a.

Vertex algebras with an action of sl_2 as above were called *quasi-vertex operator algebras* in [5] and minimal elements are sometimes called *quasi-primary*.

If A has a Virasoro element ω, then we always have $D = \omega(0)$ and $\delta = \omega(1)$, and we can take $D^* = \omega(2)$.

Throughout this paper we will always assume that all vertex algebras have an sl_2-structure, and all ideals, homomorphisms, etc. must agree with this structure. For example, we call a vertex algebra *simple* if it does not have D^*-invariant ideals. Note that any D^*-invariant ideal is homogeneous, because it must be stable under the grading derivation δ.

In the sequel we will need the following easy property of sl_2-modules of the above type:

Proposition 1. *Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded module over the Lie algebra $\mathrm{sl}_2 = \Bbbk D + \Bbbk\delta + \Bbbk D^*$, where $\deg D = 1$, $\deg D^* = -1$, D^* is locally nilpotent, $\delta|_{M_d} = d$ and the commutation relations (6) hold. Then $M_d = (D^*)^{1-d} M_1$ for any $d < 0$.*

Proof. Let us first prove that $M_d = D^* M_{d+1}$ for $d < 0$. Take some element $a \in M_d$ of $\operatorname{ord} a = k$. Set

$$b = D^*Da - (k+1)(2d-k)a = DD^*a - k(2d-k-1)a \in M_d.$$

If a is minimal, then $b=0$. Otherwise, using that $(D^*)^{k+1}Da=(k+1)(2d-k)(D^*)^k a$, we get $(D^*)^k b=0$, therefore $\operatorname{ord} b<k$. By induction we can assume that $b\in D^*M_{d+1}$, and hence also $a\in D^*M_{d+1}$, because $(k+1)(2d-k)\neq 0$ for $d<0$ and $k\geqslant 0$.

Now the proposition follows from the following claim:

$$M_0=D^*M_1+\operatorname{Ker} D^*.$$

Take an element $a\in M_0$ of $\operatorname{ord} a=k$. We will prove by induction on k that $a\in D^*M_1+\operatorname{Ker} D^*$. If $k=0$ then $a\in \operatorname{Ker} D^*$. Otherwise, take as before $b=D^*Da+k(k+1)a$ so that $\operatorname{ord} b<k$. By induction, $b\in D^*M_1+\operatorname{Ker} D^*$, and therefore also $a\in D^*M_1+\operatorname{Ker} D^*$, since $k(k+1)\neq 0$ for $k>0$. □

It follows from Proposition 1 that $DM_{-1}\subset D^*M_1$. Indeed, if $a\in M_{-1}$, then $a=D^*b$ for some $b\in M_0$, and we have $Da=DD^*b=D^*Db$.

For the case when $M=A$ is a vertex algebra we will obtain that $A_d=D^*A_{d+1}$ for $d<0$ as a corollary of Lemma 1 in Section 3.

Remark. The action of sl_2 on vertex algebras was investigated in [4]. For $d\geqslant 0$, it is no longer true that $A_d=D^*A_{d+1}$. We shall see in Section 3 that the dual space to A_0/D^*A_1 parameterizes the invariant bilinear forms on A.

1.3. Modules over a vertex algebra

Let $M=\bigoplus_{d\in\mathbb{Z}} M_d$ be a graded module over $\mathrm{sl}_2=\langle D,\delta,D^*\rangle$ as in Proposition 1. A *vertex operator* on M is a formal series

$$\alpha(z)=\sum_{n\in\mathbb{Z}}\alpha(n)z^{-n-1},\qquad \alpha(n)\in\operatorname{End} M,$$

such that $[D,\alpha(z)]=\partial_z\alpha(z)$ and for any $v\in M$ we have $\alpha(n)v=0$ for $n\gg 0$. We say that $\deg\alpha(z)=d$ if $\alpha(n)M_m\subseteq M_{m+d-n-1}$. We define the space $\mathrm{vo}(M)\subset\operatorname{Hom}(M,M((z)))$ to be the span of all homogeneous vertex operators.

Define the action of D^* on $\mathrm{vo}(M)$ by

$$(D^*\alpha)(n)=[D^*,\alpha(n)]-(2d-n-2)\alpha(n+1)$$

for $\alpha\in\mathrm{vo}(M)$ of degree d. For a pair of vertex operators $\alpha,\beta\in\mathrm{vo}(M)$ define their product $\alpha(m)\beta\in\mathrm{vo}(M)$ by

$$(\alpha(m)\beta)(n)=\sum_{s\geqslant 0}(-1)^s\binom{m}{s}\alpha(m-s)\beta(n+s)$$
$$-\sum_{s\leqslant m}(-1)^s\binom{m}{m-s}\beta(n+s)\alpha(m-s).$$

Definition 2. A module over a vertex algebra A is a graded sl_2-module M with a homomorphism $\pi:A\to\mathrm{vo}(M)$, that preserves the products, degree and commutes with the action of sl_2.

The associativity formula (4) means that the left adjoint action map $Y: A \to \mathrm{vo}(A)$ satisfies $Y(a(n)b) = Y(a)(n)Y(b)$, and (iii) of Definition 1 together with (5) imply that Y is an sl_2-module homomorphism, therefore a vertex algebra is always a module over itself.

Definition 2 is due to Li [13]. Equivalently, one can define a module over a vertex algebra A as a graded sl_2-module M with an action $a(m): M \to M$ for any $a \in A$ and $m \in \mathbb{Z}$, so that the sum $A \oplus M$ is a vertex algebra with $\mathbb{1} = (\mathbb{1}, 0)$, $D = (D, D)$ and the products

$$(a,u)(n)(b,v) = \left(a(n)b,\, a(n)v - \sum_{i \geqslant 0} (-1)^{n+i} D^{(i)}(b(n+i)u) \right).$$

Note that we have used the quasi-symmetry formula (2).

2. The universal enveloping algebra

2.1. The construction

For any vertex algebra A we can construct a Lie algebra $L = \mathrm{Coeff}\, A$ in the following way [1,10,14,15]. Consider the linear space $\mathbb{k}[t,t^{-1}] \otimes A$, where t is a formal variable. Denote $a(n) = a \otimes t^n$ for $n \in \mathbb{Z}$. As a linear space, L the quotient of $\mathbb{k}[t,t^{-1}] \otimes A$ by the subspace spanned by the relations

$$(Da)(n) = -na(n-1). \tag{7}$$

The brackets are given by

$$[a(m), b(n)] = \sum_{i \geqslant 0} \binom{m}{i} (a(i)b)(m+n-i) \tag{8}$$

which is precisely identity (iv) of Definition 1. The spaces $L_\pm = \mathrm{Span}\{a(n) \mid n \gtreqless 0\} \subset L$ are Lie subalgebras of L and we have $L = L_- \oplus L_+$.

Remark. The construction of L makes use of only the products (n) for $n \geqslant 0$ and the map D. This means that it works for a more generic algebraic structure, known as *conformal algebra* [10].

Now assume that the vertex algebra A has the sl_2-structure. Then formulas (1) and (5) define derivations $D: L \to L$ and $D^*: L \to L$ so we get an action of sl_2 on L by derivations. Denote by $\hat{L} = L \rtimes \mathrm{sl}_2$ the corresponding semi-direct product.

The Lie algebra $\hat{L} = \widehat{\mathrm{Coeff}}\, A$ and its universal enveloping algebra $U = U(\hat{L})$ inherit the grading from A so that $\deg a(m) = \deg a - m - 1$. The *Frenkel–Zhu topology* [7] on a homogeneous component U_d is defined by setting the neighborhoods of 0 to be the spaces $U_d^k = \sum_{i \leqslant k} U_{d-i} U_i$, so that

$$\cdots \subset U_d^{k-1} \subset U_d^k \subset U_d^{k+1} \subset \cdots \subset U_d, \qquad \bigcap_{k \in \mathbb{Z}} U_d^k = 0, \qquad \bigcup_{k \in \mathbb{Z}} U_d^k = U_d.$$

Let $\bar{U} = \bigoplus_{d\in\mathbb{Z}} \bar{U}_d$ be the completion of $U(\hat{L})$ in this topology. Consider the ideal $I \subset \bar{U}$ generated by the relations

$$\sum_{s\geqslant 0} \binom{k}{s} (a(m+s)b)(n+k-s)$$
$$= \sum_{s\geqslant 0} (-1)^s \binom{m}{s} a(m+k-s)b(n+s)$$
$$- \sum_{s\leqslant m} (-1)^s \binom{m}{m-s} b(n+s)a(m+k-s) \tag{9}$$

for all $a, b \in A$ and $k, m, n \in \mathbb{Z}$. Note that the relations above are simply the Borcherds identity (3). Denote by $W = \bar{U}/\bar{I}$ the quotient of $\bar{U}$ by the closure of I.

Remark. In fact, the ideal $I \subset \bar{U}$ is generated only by relation (9) for $k=0$ and $n < 0$. We will not need this fact.

Let $\mathscr{G} \subset A$ be a set of generators of the vertex algebra A. We claim that the set of coefficients $\{a(m) \mid a \in \mathscr{G},\ m \in \mathbb{Z}\}$ topologically generates W. Indeed, any element $b \in A$ can be written as linear combination of vertex monomials $a_1(m_1)\cdots a_l(m_l)a_{l+1}$, with some order of parentheses, where $a_i \in \mathscr{G}$ and $m_i \in \mathbb{Z}$. Then, using associativity (4), we can write a coefficient $b(n)$ as a (possibly infinite) combination of associative words $a_{i_1}(n_1)\cdots a_{i_l}(n_l)a_{i_{l+1}}(n_{l+1})$.

For a finite ordered set of generators $\mathscr{S} = \{a_1, \ldots, a_l\}$, $a_i \in \mathscr{G}$, let $W_{\mathscr{S}}$ be the $\langle D, \delta, D^* \rangle$-module generated by all monomials $a_1(m_1)\cdots a_l(m_l) \in W$, $m_i \in \mathbb{Z}$.

Definition 3. The universal enveloping algebra of A is

$$U(A) = \bigcup_{\mathscr{S}} \overline{W}_{\mathscr{S}} \subset W,$$

where the union is taken over all finite ordered sets of generators of $\mathscr{G}$, and $\overline{W}_{\mathscr{S}}$ is the completion of the space $W_{\mathscr{S}}$ in the Frenkel–Zhu topology.

Remark. In fact, one can show that if $\mathscr{S}$ and $\mathscr{S}'$ differ by a permutation, then $\overline{W}_{\mathscr{S}} = \overline{W}_{\mathscr{S}'}$.

Proposition 2. *Any module over a vertex algebra A is a continuous module over $U(A)$, in the sense that for any sequence $u_1, u_2, \ldots \in U(A)$ that converges to 0 and any $v \in M$ we have $u_i v = 0$ for $i \gg 0$. Conversely, any $U(A)$-module M, such that for any $a \in A$ and $v \in M$ one has $a(m)v = 0$ for $m \gg 0$, is a module over A.*

Proof. Take a finite set $\mathscr{S} = \{a_1, \ldots, a_l\}$, $a_i \in \mathscr{G}$ as above. Let $u_1, u_2, \ldots \in W_{\mathscr{S}}$ be a sequence of homogeneous elements of the same degree, such that $\lim_{i\to\infty} u_i = 0$. This means that u_i is a span of elements of the form $u_i' u_i''$, where $u_i' \in W_{\mathscr{S}'}$, $u_i'' \in W_{\mathscr{S}''}$ for $\mathscr{S} = \mathscr{S}' \sqcup \mathscr{S}''$ and $\lim_{i\to\infty} \deg u_i'' = -\infty$.

Let $\hat{\mathscr{S}}=\{(D^*)^j a \mid a\in\mathscr{S},\ j\geqslant 0\}$. Since $\mathscr{S}$ is finite and D^* is locally nilpotent, the set $\hat{\mathscr{S}}$ is also finite. The space $W_{\mathscr{S}}$ is spanned over $\Bbbk[D]$ by monomials $b_1(m_1)\cdots b_l(m_l)$, for $b_i\in\hat{\mathscr{S}}$ and $m_i\in\mathbb{Z}$. Let $N=\max\{N(a,b)\mid a,b\in\hat{\mathscr{S}}\}$. By the results of [15], the space $W_{\mathscr{S}}$ is spanned over $\Bbbk[D]$ by the monomials $b_1(m_1)\cdots b_l(m_l)$ such that $m_i-m_{i+1}\leqslant N$ for all $1\leqslant i\leqslant l-1$. Let $d=\min\{\deg b\mid b\in\hat{\mathscr{S}}\}$. For any $v\in M$, there is $n\in\mathbb{Z}$, such that $b(m)v=0$ for all $b\in\hat{\mathscr{S}}$ and $m\geqslant n$. It follows, that if an element $u\in W_{\mathscr{S}''}$ has degree less than

$$\deg(b_1(n+(l-1)N)\cdots b_{l-1}(n+N)b_l(n))\geqslant dl-l(n+1)-\tfrac{1}{2}\,l(l-1)N$$

then $uv=0$.

Conversely, let M be a module over $U(A)$ with the above property. Define a map $\pi:A\to \mathrm{vo}(M)$ by $\pi(a)=\sum_{m\in\mathbb{Z}} a(m)z^{-m-1}$. Relation (9) for k=0 imply that $\pi(a(n)b)=\pi(a)(n)\pi(b)$, and we have defined the derivations D,D^* and δ on L so that π commutes with the action of sl_2. $\square$

Remark. The algebra $W=\bar{U}(\hat{L})/\bar{I}$ is also a good candidate for universal enveloping algebra of A. It has the following property, analogous to Proposition 2 (see [7]): consider a graded space M such that $M_d=0$ for $d\ll 0$; then M is a W-module if and only if M is an A-module.

On the other hand, we could define an algebra $\hat{U}(A)$ such that any series of elements from $U(\hat{L})$, that make sense as an operator on any A-module, would converge in $\hat{U}(A)$. However, this algebra is too big for our purposes, for example Proposition 3 below would fail for this algebra.

2.2. *The involution on the universal enveloping algebra*

Let A be a graded vertex algebra with sl_2-structure, as in Section 1.2.

Proposition 3. *Set*

$$a(m)^*=(-1)^{\deg a}\sum_{i\geqslant 0}(D^{*(i)}a)(2\deg a-m-i-2) \tag{10}$$

for a homogeneous $a\in A$ and $m\in\mathbb{Z}$, and also $D^{**}=D$ and $\delta^*=\delta$. Then there is a continuous anti-involution on the enveloping algebra $U(A)$ defined by $x\mapsto x^*$.

Note that $U(A)^*_d\subseteq U(A)_{-d}$.

Remark. The computations of Steps 1–3 in the proof below were done in [5], using the techniques of formal calculus. We thank the referee for pointing this out. For the sake of completeness though, we present these computations using our notations.

For $a\in A$, denote $a^{(i)}=D^{*(i)}a$.

Proof of Proposition 3. *Step* 1: First we check that $a(m)^{**}=a(m)$. Let $\deg a=d$. Then

$$a(m)^{**}=(-1)^d\sum_{i\geqslant 0}a^{(i)}(2d-m-i-2)^*$$

$$=\sum_{i,j\geqslant 0}\binom{i+j}{i}a^{(i+j)}(m-i-j)$$

$$=\sum_{j\geqslant 0}\sum_{i\geqslant 0}(-1)^i\binom{j}{i}a^{(j)}(m-j)=a(m).$$

Step 2: Next we show that our involution preserves relation (7). This amounts to check that

$$[a(m)^*,D^*]=-ma(m-1)^*.$$

Indeed, using (5) and (10), we compute

$$[D^*,a(m)^*]=(-1)^d\sum_{i\geqslant i}[D^*,a^{(i)}(2d-m-i-2)]$$

$$=(-1)^d\sum_{i\geqslant 0}((m-i)a^{(i)}(2d-m-i-1)$$

$$+(i+1)a^{(i+1)}(2d-m-i-2))$$

$$=(-1)^d m\sum_{i\geqslant 0}a^{(i)}(2d-m-i-1)=ma(m-1)^*.$$

Step 3: Now we are going to check that our involution preserves relation (9). Take $a\in A_d$, $b\in A_e$ and $k,m,n\in\mathbb{Z}$, and denote the corresponding relation (9) by $R_{a,b}(k,m,n)$.

By iterating formula (5) we get

$$(a(s)b)^{(t)}=\sum_{i,j\geqslant 0}\binom{2d-s-i-2}{j}a^{(i)}(s+j)b^{(t-i-j)}.$$

Using this, we apply * to the left-hand side of $R_{a,b}(k,m,n)$ and get

$$\sum_s\binom{k}{s}(a(m+s)b)(n+k-s)^*$$

$$=(-1)^{d+e+m+1}\sum_{s,t}(-1)^s\binom{k}{s}(a(m+s)b)^{(t)}$$

$$\times(2(d+e-m-s-1)-n-k+s-t-2)$$

$$=(-1)^{d+e+m+1}\sum_{i,j,s,t}(-1)^s\binom{k}{s}\binom{2d-m-s-i-2}{j}$$

$$\times(a^{(i)}(m+s+j)b^{(t-i-j)})(2(d+e-m)-k-n-s-t-4)$$

$$=(-1)^{d+e+m+1}\sum_{i,j,s}\binom{2d-k-m-i-2}{s}$$
$$\times(a^{(i)}(m+s)b^{(j)})(2(d+e-m)-k-n-i-j-s-4)$$

which is the left-hand side of

$$(-1)^{d+e+m+1}\sum_{i,j}R_{a^{(i)},b^{(j)}}(2d-k-m-i-2,\, m,\, 2e-m-n-j-2). \tag{11}$$

We have used here the identity

$$\sum_{t=0}^{s}(-1)^t\binom{k}{t}\binom{r-t}{s-t}=\binom{r-k}{s}.$$

On the other hand, applying the involution to the right-hand side of $R_{a,b}(k,m,n)$, we get

$$\sum_{i,j,s}(-1)^{d+e+s}\binom{m}{s}b^{(j)}(2e-n-s-j-2)a^{(i)}(2d-k-m+s-i-2)$$
$$-\sum_{i,j,s}(-1)^{d+e+s}\binom{m}{m-s}a^{(i)}(2d-k-m+s-i-2)b^{(j)}(2e-n-s-j-2)$$

which is exactly the right-hand side of (11).

Note that the commutation relations (8) are a special case of (9) for $m=0$, so our computation also shows that * preserves (8).

Step 4: Since by Step 2, the involution * preserves relations (7) and (8), it can be extended to an anti-involution of the Lie algebra $\hat{L}$, and of its universal enveloping algebra $U(\hat{L})$. It is left to show that $^*: U(\hat{L})\to U(\hat{L})$ is continuous in the Frenkel–Zhu topology (see Section 2).

Denote by U_d the component of degree d of $U(\hat{L})$. Let $u_1,u_2,\ldots\in U_d$ be a convergent sequence. This means that for any $k\in\mathbb{Z}$ all elements u_n with sufficiently large n belong to $U_d^{k+d}=\sum_{i\leqslant k+d}U_{d-i}U_i$. But then

$$u_n^*\in\sum_{i\leqslant k+d}(U_{d-i}U_i)^*\subseteq\sum_{i\leqslant k+d}U_{-i}U_{i-d}=U_{-d}^k$$

therefore the sequence $\{u_n^*\}$ also converges in the Frenkel–Zhu topology. □

3. Invariant bilinear forms

Let as before A be a vertex algebra with sl_2-structure. Let V be a vector space over $\Bbbk$.

Definition 4. A V-valued bilinear form $\langle\cdot|\cdot\rangle$ on A is called *invariant* if

$$\langle a(m)b|c\rangle=\langle b|a(m)^*c\rangle \quad\text{and}\quad \langle Da|b\rangle=\langle a|D^*b\rangle$$

for all $a,b,c\in A$ and $m\in\mathbb{Z}$.

The radical $\mathrm{Rad}\langle\cdot|\cdot\rangle = \{a\in A \mid \langle a\,|\,b\rangle = 0\ \forall b\in A\}$ of an invariant form is an ideal of A. Also, we have

$$\langle \delta a\,|\,b\rangle = \frac{1}{2}\,\langle [D^*,D]a\,|\,b\rangle = \frac{1}{2}\,\langle a\,|\,[D^*,D]b\rangle = \langle a\,|\,\delta b\rangle$$

and this implies that $\langle A_i\,|\,A_j\rangle = 0$ for $i\neq j$.

When A is a highest weight module over affine or Virasoro Lie algebra $\mathfrak{g}$ (see e.g. [7,10]), then one can choose D^* so that the involution * of Section 2.2 is the extension of the Chevalley involution of $\mathfrak{g}$, and the contravariant form on A (see e.g. [9]) will be invariant in the sense of Definition 4. See also the example in Section 5.1.

Given an invariant form $\langle\cdot|\cdot\rangle$ on A, one can consider a linear map $f: A_0\to V$ defined by $f(a) = \langle \mathbb{1}\,|\,a\rangle$. Clearly, $f(D^*A_1) = 0$, so that $f\in \mathrm{Hom}(A_0/D^*A_1, V)$, and f defines the form uniquely. We show that this in fact gives an isomorphism of the space of all invariant bilinear forms with $\mathrm{Hom}(A_0/D^*A_1, V)$.

Theorem 1. *Every invariant bilinear form on A is symmetric and there is a one-to-one correspondence between linear maps $f: A_0/D^*A_1\to V$ and V-valued invariant bilinear forms $\langle\cdot|\cdot\rangle$ on A, given by $f(a) = \langle \mathbb{1}\,|\,a\rangle$ for $a\in A_0$.*

For the proof we need the following lemma.

Lemma 1. *For any $a, b\in A$, if either $m\geqslant 0$ or $m < -\operatorname{ord} b - 1$, then $a(m)^*b\in D^*A$.*

Proof. Set $\deg a = d$ and $\operatorname{ord} b = k$. Denote $((D^*)^j a)(l) = g^{d-j-l-1}_{2(d-j)-l-1}\in U(A)$, so that

$$g^n_m = ((D^*)^{d+n-m}a)(m-2n-1).$$

For an operator $u\in U(A)$ write $u\sim 0$ if $ub\in D^*A$. Then by (5) we have

$$m g^n_m + g^n_{m-1} + g^{n+1}_{m+1}D^* \sim 0.$$

Consider the generating function $g(x,y) = \sum_{m,n\in\mathbb{Z}} g^n_m x^m y^n$. Let $R: U(A)\to U(A)$ be the operator given by $Ru = uD^*$. Note that $R^i\sim 0$ for $i>k$, since $(D^*)^{k+1}b = 0$. Then the above relation reads as

$$x\partial_x g + xg + \frac{R}{xy}\,g \sim 0$$

therefore

$$g(x,y) \sim \exp\left(-x + \frac{R}{xy}\right) g_0(y)$$

for some series $g_0(y)\in U(A)[[y^{\pm 1}]]$. On the other hand,

$$a(m)^* = (-1)^d \sum_{i\geqslant 0} (D^{*(i)}a)(2d-m-2-i) = (-1)^d \sum_{i\geqslant 0} \frac{1}{i!}\, g^{m+1-d}_{m+1-i}.$$

Set $h^n_m = \sum_{i\geqslant 0} \frac{1}{i!}\, g^n_{m-i}$, so that $a(m)^* = (-1)^d h^{m+1-d}_{m+1}$. The generating function for h^n_m's is

$$h(x,y) = \sum_{m,n\in\mathbb{Z}} h^n_m x^m y^n = \mathrm{e}^x g(x,y) \sim \exp\left(\frac{R}{xy}\right) g_0(y).$$

Since $(R)^i \sim 0$ for $i > k$, the only powers of x with coefficients $\nsim 0$ in $h(x, y)$ are x^0, $x^{-1}, \dots, x^{-k}$, therefore $h^n_m \sim 0$ if either $m > 0$ or $m < -k$, and the lemma follows. □

Remark. As a corollary, we get a proof of the fact that $A_d = D^* A_{d+1}$ different from that of Proposition 1. Indeed, take an element $a \in A_d$, $d < 0$. Write $a = a(-1)\mathbb{1}$. Then $a(-1)$ is the dual of $\pm \sum_{i \geqslant 0} (D^{*(i)} a)(2d - i - 1)$. But $\operatorname{ord} \mathbb{1} = 0$ and $2d - i - 1 \neq -1$ for all $i \geqslant 0$, so Lemma 1 implies that $a(-1)\mathbb{1} \in D^* A_{d+1}$.

Proof of Theorem 1. We need to show that for every linear map $f : A_0/D^* A_1 \to V$ there is a symmetric invariant bilinear form $\langle \cdot | \cdot \rangle$ on A such that $f(a) = \langle \mathbb{1} | a \rangle$ for all $a \in A_0$.

For homogeneous $a, b \in A$, define $\langle a \,|\, b \rangle = f(a(-1)^* b)$ if $\deg a = \deg b$ and $\langle a \,|\, b \rangle = 0$ otherwise. Let us show first that $\langle a(n)b \,|\, c \rangle = \langle b \,|\, a(n)^* c \rangle$ for all $a, b, c \in A$ and $n \in \mathbb{Z}$. Indeed, it is enough to check that $(a(n)b)(-1)^* c \equiv b(-1)^* a(n)^* c \bmod D^* A$. But using associativity (4), we have $(a(n)b)(-1) - a(n)b(-1) \in U(A)L_+$, and by Lemma 1, $u^* A \subseteq D^* A$ for any $u \in U(A)L_+$. Similarly, $\langle Da \,|\, b \rangle = f(a(-1)^* D^* b) = \langle a \,|\, D^* b \rangle$. This proves that the form is invariant.

Now we show that the form $\langle \cdot | \cdot \rangle$ is symmetric. We need to have $a(-1)^* b \equiv b(-1)^* a \bmod D^* A$ for every $a, b \in A$ of the same degree. Set $u = a(-1)^* b(-1) \in U(A)$ so that $\deg u = 0$. We have to show that

$$(u - u^*)\mathbb{1} \in D^* A_1. \tag{12}$$

By (10), u is a linear combination of operators of the form $c(m)b(-1)$ for some $c \in A$ and $m \in \mathbb{Z}$. But $c(m)b(-1) \equiv (c(m)b)(-1) \bmod U(A)L_+$. Hence it is enough to prove (12) for the cases when $u \in U(A)L_+$ and when $u = a(-1)$ for some $a \in A_0$.

If $u \in U(A)L_+$ then $u\mathbb{1} = 0$ and $u^*\mathbb{1} \in D^* A$ by Lemma 1. If $u = a(-1)$ for $a \in A_0$, then

$$a(-1) - a(-1)^* = (a(-1)^* - a(-1))^* = \sum_{i \geqslant 1} (D^{*(i)} a)(-i - 1)^*$$

and again by Lemma 1 we conclude that $(a(-1) - a(-1)^*)\mathbb{1} \in D^* A_1$, since $\operatorname{ord} \mathbb{1} = 0$. □

Remark. It follows from Theorem 1 that if $u \in U(A)$, $\deg u = d$, is such that $u\mathbb{1} = 0$, then $u^* A_d \in D^* A_1$. Indeed, $0 = \langle u\mathbb{1} \,|\, A_d \rangle = \langle \mathbb{1} \,|\, u^* A_d \rangle$ for every invariant bilinear form on A. In fact, using techniques from [15,16] one can show that $\{u \in U(A) \,|\, u\mathbb{1} = 0\} = U(A)L_+$.

4. Radical of a vertex algebra

4.1. The ideal generated by $D^* A_1$

Let A be a vertex algebra with an sl_2-structure as before. Let $I \subset A$ be the ideal generated by $D^* A_1$. Denote as usual $I_d = I \cap A_d$.

Proposition 4. (a) $I_d = A_d$ *for* $d < 0$.
(b) $I_0 = A(-1)D^*A_1$.
(c) A_0/I_0 *is an associative commutative algebra with respect to the operation* $a \otimes b \mapsto ab = a(-1)b$, *and* $\mathbb{1}$ *is its unit.*

Denote $E = A(-1)D^*A_1$. For the proof of Proposition 4 we need the following lemma.

Lemma 2. *Let* $a \in A_i$ *and* $b \in A_j$ *for* $i, j \leqslant 0$, *and if* $j = 0$, *then* $b \in D^*A_1$. *Then* $a(i+j-1)b \in E$.

Proof. By Proposition 1 we have $a = (D^*)^i a_0$ and $b = (D^*)^j b_0$ for some $a_0 \in A_0$ and $b_0 \in D^*A_1$. Applying (5) $-i$ times we get that

$$a(i+j-1)b \equiv (-1)^i a_0(i+j-1)(D^*)^{-i-j} b_0 \bmod E.$$

If $d = i + j = 0$, then $a(i+j-1)b = a(-1)b \in E$ by the definition. We will show by induction on d that $a(d-1)(D^*)^{-d} b \in E$ for any $a \in A_0$, $b \in D^*A_1$ and $d < 0$.

So assume that $a(k-1)(D^*)^{-k} b \in E$ for all $0 \geqslant k > d$. Then, using (1), we get

$$E \supset D^*A_1 \ni Da(d)(D^*)^{-d} b = a(d)D(D^*)^{-d} b - da(d-1)(D^*)^{-d} b.$$

By Proposition 1, $D(D^*)^{-d} b = (D^*)^{1-d} b_1$ for some $b_1 \in D^*A_1$, so by induction, $a(d)D(D^*)^{-d} b \in E$, and hence also $a(d-1)(D^*)^{-d} b \in E$. □

Remark. In fact, Lemma 2, combined with Lemma 1, shows that $E = U(A)_0 D^*A_1$.

Proof of Proposition 4. (a) Follows immediately from Proposition 1.

(b) Let $f : A_0 \to A_0/E$ be the canonical projection, and let $R = \mathrm{Rad}\langle \cdot | \cdot \rangle$ be the radical of the corresponding form on A. We claim that $R_0 = R \cap A_0 = E$. Indeed, if $a \in A_0 \setminus E$, then $f(a) = \langle a \,|\, \mathbb{1} \rangle \neq 0$ and therefore $a \notin R$. In the other direction, take $a \in D^*A_1$. Then for an arbitrary $b \in A_0$ we have

$$\langle a \,|\, b \rangle = f(b(-1)^* a) = \sum_{i \geqslant 0} f((D^{*(i)} b)(-1-i)a) = 0$$

since $(D^{*(i)} b)(-1-i)a \in E$ by Lemma 2. Therefore, $D^*A_1 \subset R$, and since R is an ideal, we must have $E \subset R$. It follows that $I \subset R_0 = E$, on the other hand obviously $E \in I$, and that proves (b).

(c) Using (4) and Lemma 2 we get

$$\begin{aligned}(a(-1)b)(-1)c &= \sum_{s \geqslant 0} a(-1-s)b(-1+s)c + \sum_{s \geqslant 0} b(-2-s)a(s)c \\ &\equiv a(-1)b(-1)c \bmod I_0\end{aligned}$$

for any $a, b, c \in A_0$. By (2) we have $a(-1)b \equiv b(-1)a \bmod DA_{-1}$, and we know that $DA_{-1} \subset D^*A_{i+1} \subset I_0 = E$. So the statement (c) follows. □

4.2. The radical

Definition 5. The radical $\operatorname{Rad} A$ of a vertex algebra A is the radical $\operatorname{Rad}\langle\cdot|\cdot\rangle$ of the invariant bilinear form $\langle\cdot|\cdot\rangle$ corresponding to the projection $f: A_0 \to A_0/A_0(-1)D^*A_1$.

Remark. This definition has nothing to do with the radical defined in [3].

Recall that by Proposition 4(b), the space $I_0 = A_0(-1)D^*A_1$ is the degree 0 component of the ideal $I \subset A$ generated by D^*A_1. From the proof of Proposition 4(b) it follows that $(\operatorname{Rad} A)_0 = I_0$. Moreover, the following is true:

Proposition 5. (a) *The radical $R = \operatorname{Rad} A$ is the maximal among all the ideals $J \subset A$ such that $J_0 = A_0(-1)D^*A_1$.*

(b) $\operatorname{Rad}(A/R) = 0$.

Proof. Let $J \subset A$ be an ideal such that $J_0 = A_0(-1)D^*A_1$. Then for any $a \in J_d$ and $b \in A_d$ we get $\langle a \mid b\rangle = f(b(-1)^*a) = 0$, therefore $a \in R$. This proves (a). Statement (b) follows from the fact that the form $\langle\cdot|\cdot\rangle$ becomes non-degenerate on A/R. $\square$

4.3. Radical-free algebras

Assume that $R = \operatorname{Rad} A \subsetneq A$. We would like to investigate the quotient vertex algebra A/R. By Proposition 5(b) we have $\operatorname{Rad} A/R = 0$, so we can assume that $\operatorname{Rad} A = 0$. Then $D^*A_1 = I = 0$, the projection $f: A_0 \to A_0/I_0$ is the identity map, and the corresponding canonical invariant A_0-valued bilinear form $\langle\cdot|\cdot\rangle$ is non-degenerate.

The following theorem summarizes the properties of radical-free algebras.

Theorem 2. *Let A be a vertex algebra such that* $\operatorname{Rad} A = 0$.

(a) $A_d = 0$ *for* $d < 0$.
(b) A_0 *is an associative commutative unital algebra with respect to the product* $a \otimes b \mapsto ab = a(-1)b$, *with unit* $\mathbb{1}$.
(c) *A is a vertex algebra over the associative commutative algebra A_0.*
(d) *Let $I \subset A$ be an ideal. Then $I \cap A_0 = 0$ if and only if $I = 0$.*
(e) *For any ideal $J_0 \subset A_0$ the space $J_0 A$ is an ideal of A such that $J_0 A \cap A_0 = J_0$. It follows that A is simple if and only if A_0 is a field.*

Note that all the definitions in Section 1 make sense when $\Bbbk$ is a commutative ring containing $\mathbb{Q}$. So statement (c) means that elements from $A_0 \subset A$ behave like constants, i.e. A_0 acts on each component A_d by $ab = a(-1)b$ for $a \in A_0$, $b \in A_d$, all structural maps $a(m), D, D^*: A \to A$ are A_0-linear, and therefore $DA_0 = 0$ and $A_0(n)A_d = 0$ for $n \neq -1$. Also, the canonical non-degenerate A_0-valued form $\langle\cdot|\cdot\rangle$ on A is A_0-bilinear.

Recall that a vertex algebra with sl_2-structure is called simple if it does not have any D^*-invariant ideals.

Proof. Statements (a) and (b) follow from Proposition 4(a) and (c), respectively, while (d) follows from Proposition 5(a).

Since $D^*A_1 = 0$, we have $\langle Da \mid b\rangle = 0$ for any and $a \in A_0$, $b \in A_1$, therefore $DA_0 \subset \operatorname{Rad} A = 0$. But then $a(n) = 0$ for $n \neq -1$ as an element in the Lie algebra $\operatorname{Coeff} A$, see Section 2. Now the statement (c) follows easily from formulas (1), (4) and (5) in Section 1.1.

Finally, $I_0 A$ is an ideal of A by (c), and therefore, if A is simple, then A_0 must be a field. This proves one direction of (e). The other direction follows from (d). □

Remark. Let A be a vertex algebra with $\operatorname{Rad} A = 0$ as above. Since $DA_0 = 0$, formulas (4) and (2) imply that A_1 is a Lie algebra with the bracket $[a, b] = a(0)b$. The bilinear form $\langle\cdot|\cdot\rangle$ on A is an invariant symmetric A_0-bilinear form on that Lie algebra, and we have $\langle a \mid b\rangle = a(1)b$ for any $a, b \in A_1$. The Lie algebra A_1 acts on A by derivations of degree 0, so that $ab = a(0)b$ for $a \in A_1$ and $b \in A$. If A is simple, then $A_1(0)A = 0$, since $A_1(0)A$ is always an ideal of A and $\mathbb{1} \notin A_1(0)A$.

If $\Bbbk = \mathbb{C}$ and A is a simple radical-free vertex algebra, generated by countably many elements, then we must have $A_0 = \mathbb{C}$. We see that radical-free vertex algebras are very close to be of a CFT type, after we set $\Bbbk = A_0$. The only thing that might be missing is the condition $\dim A_d < \infty$.

Question. Are there any radical-free vertex algebras A with sl_2-structure, such that the homogeneous components A_d have infinite rank over A_0?

5. Examples

5.1. Heisenberg algebra

Let A be the Heisenberg vertex algebra. By definition, it is generated by a single element $a \in A_1$ such that the locality $N(a, a) = 2$ and $a(0)a = 0$, $a(1)a = \mathbb{1}$. One can show (see e.g. [16]) that these conditions define A uniquely. We have $A_0 = \Bbbk\,\mathbb{1}$ and $A_1 = \Bbbk a$. It is well known, that A is a simple vertex algebra in the sense that it does not have any non-trivial ideals, regardless of any sl_2-structure.

For every $k \in \Bbbk$ the element $\omega_k = \frac{1}{2}\, a(-1)a + kDa \in A_2$ is a Virasoro element of A. If we set $D^* = \omega_0(2)$, then $D^*a = 0$, hence $A_0/D^*A_1 = \Bbbk\,\mathbb{1}$, and therefore A has only one invariant bilinear form $\langle\cdot|\cdot\rangle$, up to a scaling factor, and in this case $\operatorname{Rad} A = \operatorname{Rad}\langle\cdot|\cdot\rangle = 0$. If instead we set $D^* = \omega_k(2)$ for $k \neq 0$, then $D^*a = -2k\mathbb{1}$. In this case $A_0/D^*A_1 = 0$, $\operatorname{Rad} A = A$, and A does not have any invariant bilinear forms.

5.2. Free vertex algebras

Let $\mathscr{G}$ be a set and $N : \mathscr{G} \times \mathscr{G} \to \mathbb{Z}$ be a symmetric function such that $N(a, a) \in 2\mathbb{Z}$. In [16] we have constructed a vertex algebra $F = F_N(\mathscr{G})$ which we called *the free vertex algebra generated by* $\mathscr{G}$ *with respect to locality bound* N. It is generated by $\mathscr{G}$ so that the locality of any $a, b \in \mathscr{G}$ is exactly $N(a, b)$, and for any other vertex algebra A generated by $\mathscr{G}$ with the same bound on locality, there is a homomorphism $F \to A$ preserving $\mathscr{G}$. The algebra F is graded by the semilattice $\mathbb{Z}_+[\mathscr{G}]$, we refer to this grading as the grading by weights. Also, F is graded by degrees if we set $\deg a = -\frac{1}{2} N(a,a)$ for $a \in \mathscr{G}$. If we denote by $F_{\lambda,d}$ the space of all elements of F of weight $\lambda \in \mathbb{Z}_+[\mathscr{G}]$ and degree $d \in \mathbb{Z}$, then

$$F_{\lambda,k}(n)F_{\mu,l} \subseteq F_{\lambda+\mu,k+l-n-1}, \quad DF_{\lambda,k} \subseteq F_{\lambda,k-1} \quad \text{and} \quad \mathbb{1} \in F_{0,0}.$$

It is not difficult to see that we can define an operator $D^* : F \to F$, satisfying (5), by setting $D^*a = 0$ for all $a \in \mathscr{G}$. Then we have $D^*F_{\lambda,d} \subseteq F_{\lambda,d+1}$.

It is shown in [16] that for any weight $\lambda = a_1 + \cdots + a_l \in \mathbb{Z}_+[\mathscr{G}]$ there is the minimal degree

$$d_{\min}(\lambda) = -\frac{1}{2}\sum_{i,j=1}^{l} N(a_i,a_j) \in \mathbb{Z}$$

such that $\dim_{\Bbbk} F_{\lambda,d_{\min}(\lambda)} = 1$ and $\dim_{\Bbbk} F_{\lambda,d} = 0$ for $d < d_{\min}(\lambda)$. For $d > d_{\min}(\lambda)$ the dimension of $F_{\lambda,d}$ is equal to the number of partitions of $d - d_{\min}(\lambda)$ into a sum of l non-negative integers colored by $a_1, \ldots, a_l$.

It is also proved in [16] that F is embedded into the vertex algebra V_Λ corresponding to the lattice $\Lambda = \mathbb{Z}[\mathscr{G}]$. The scalar product on Λ is defined by $\langle a \,|\, b \rangle = -N(a,b)$ for $a, b \in \mathscr{G}$.

Since D^* is homogeneous with respect to both weights and degrees, and the only element of weight 0 in F is $\mathbb{1}$, up to a scalar, it follows that $\mathbb{1} \notin D^*F_1$. So one can always define a functional $f : F_0/D^*F_1 \to \Bbbk$ so that $f(\mathbb{1}) = 1$ and $f(F_{\lambda,0}) = 0$ for every weight $\lambda \neq 0$. The corresponding invariant bilinear form on F is a restriction of the canonical bilinear form on V_Λ. If $N(\mathscr{G},\mathscr{G}) < 0$, then this will be the only form on F, up to a scalar. If, on the contrary, the localities $N(a_i,a_j)$ are large, then $\dim F_{\lambda,0}$ grows as λ gets longer, and it follows from the results of [16] that $\dim F_{\lambda,0}/(\operatorname{Rad} F)_{\lambda,0} > 0$ when $\lambda \in \mathbb{Z}_+[\mathscr{G}]$ is big enough. Hence, in this case there are infinitely many different invariant bilinear forms on F, and infinitely many simple radical-free vertex algebras, which are homomorphic images of F.

Remark. It appears that the algebra $\bar{F} = F/\operatorname{Rad} F$ is a very interesting object from combinatorial point of view. It is possible to show that the commutative associative algebra $\bar{F}_0$ is isomorphic to a polynomial algebra in infinitely many variables. Also, the Question at the end of Section 4 is equivalent to asking whether the components $\bar{F}_d$ have finite rank as modules over $\bar{F}_0$.

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